

Competition Numbers, Quasi-Line Graphs and Holes

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Abstract

The competition graph of a directed acyclic graph D is the undirected graph on the same vertex set as D in which two distinct vertices are adjacent if they have a common out-neighbor in D . The competition number of an undirected graph G is the least number of isolated vertices that have to be added to G to make it the competition graph of a directed acyclic graph. We resolve two conjectures concerning competition graphs. First we prove a conjecture of Opsut by showing that the competition number of every quasi-line graph is at most 2. Recall that a quasi-line graph, also called a locally co-bipartite graph, is a graph for which the neighborhood of every vertex can partitioned into at most two cliques. To prove this conjecture we devise an alternative characterization of quasi-line graphs to the one by Chudnovsky and Seymour. Second, we prove a conjecture of Kim by showing that the competition number of any graph is at most one greater than the number of holes in the graph. Our methods also allow us to prove a strengthened form of this conjecture recently proposed by Kim, Lee, Park and Sano, showing that the competition number of any graph is at most one greater than the dimension of the subspace of the cycle space spanned by the holes.

Keywords: competition numbers, quasi-line graphs, characterization, holes.

1 Introduction

The *competition graph* of a directed acyclic graph D is the undirected graph on the same vertex set as D in which two distinct vertices u and v are adjacent if there is a vertex w such that (u, w) and (v, w) are arcs in D . That is, two vertices in the competition graph are adjacent, if they have a common out-neighbor in D . Competition graphs were introduced by Cohen [4] in the context of food webs, where adjacency of two vertices models the fact that they share common prey and thus compete for food.

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Roberts [16] observed that by adding a sufficient number of isolated vertices, every undirected graph G can be turned into the competition graph of some directed acyclic graph D . Quantifying this, for an undirected graph G , the *competition number* of G , denoted $\text{cn}(G)$, is the least number of isolated vertices that have to be added to G to make it the competition graph of a directed acyclic graph D .

Opsut [15] showed that computing the competition number is an NP-hard problem. He also showed that the competition number of line graphs is at most 2. He then conjectured that the bound also holds for quasi-line graphs, i.e., for graphs for which the neighborhood of every vertex can be partitioned into at most two cliques.

Conjecture 1 (Opsut [15]). If G is a quasi-line graph, then G has competition number at most 2.

Another way to bound the competition number is to consider the number of holes a graph contains. Recall that a hole is an induced cycle of length at least 4. In this context Kim [8] conjectured the following:

Conjecture 2 (Kim [8]). If a graph has at most k holes, then it has competition number at most $k + 1$.

The conjectures have previously been proven for various graph classes, as summarized at the end of this section. However, both conjectures have remained open until now.

Our results. In this paper we prove Conjectures 1 and 2. To prove Conjecture 1, we first present an alternative to Chudnovsky and Seymour's [3] structure theorem for quasi-line graphs. In the course of developing a structure theorem for the more general class of claw-free graphs, they show that every connected quasi-line graph is a fuzzy circular interval graph, or a composition of fuzzy linear interval strips.

Our characterization of quasi-line graphs, proven in Section 2, simplifies theirs.

Theorem 1. *A finite graph G is a quasi-line graph, if and only if there exists a finite graph H , a map $\phi: V(G) \rightarrow V(H)$, and a set of distinct connected subtrees $\mathbf{H} := \{T_1, \dots, T_t\}$ of H such that the following properties hold:*

1. *If two vertices v and v' of G are adjacent, then there is a tree $T_i \in \mathbf{H}$ with $\phi(v), \phi(v') \in V(T_i)$.*
2. *If two vertices v and v' of G are not adjacent and $\phi(v), \phi(v') \in V(T_i)$, then $T_i \in \mathbf{H}$ is a path with distinct endpoints $\phi(v)$ and $\phi(v')$.*
3. *If v_1 is not adjacent to v'_1 and v_2 is not adjacent to v'_2 , but $\phi(v_1) = \phi(v_2)$, $\phi(v_1), \phi(v'_1) \in V(T_i)$, and $\phi(v_2), \phi(v'_2) \in V(T_j)$, then $i = j$.*
4. *Every vertex $u \in V(H)$ of degree at least 3 is contained in at most one tree $T_i \in \mathbf{H}$.*

We call any triple (H, ϕ, \mathbf{H}) , consisting of the graph H together with the map ϕ and the subtrees $T_i \in \mathbf{H}$, a *fuzzy reconstruction* of the quasi-line graph G . Figure 1 illustrates a reconstruction of a graph G and its fuzzy vertex pairs.

Note that it is not necessarily possible to deduce G from the triple (H, ϕ, \mathbf{H}) . To easily apply fuzzy reconstructions in our proofs, we also show in Section 2 that quasi-line graphs have fuzzy reconstructions with various additional requirements. We call these pleasant reconstructions. Using fuzzy reconstructions we prove the conjecture of Opsut in Section 3.

In Section 4 we then prove Conjecture 2. Our proof method even allows us to prove a stronger form of the conjecture (Theorem 24) that recently was considered in [9].

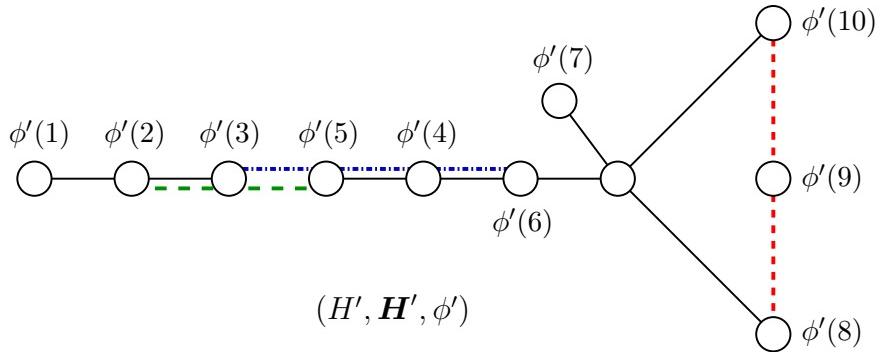
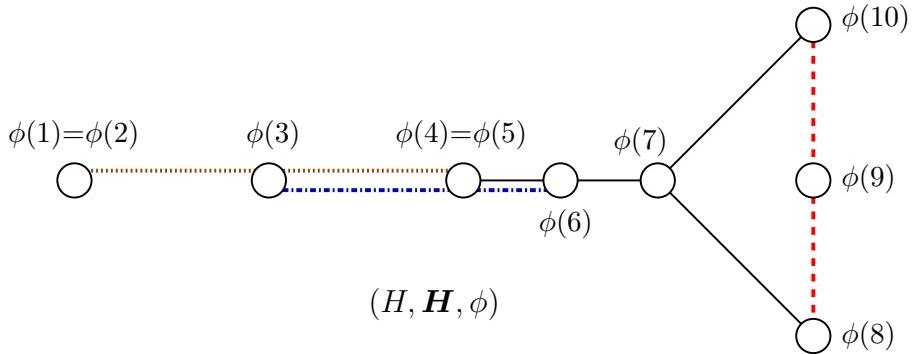
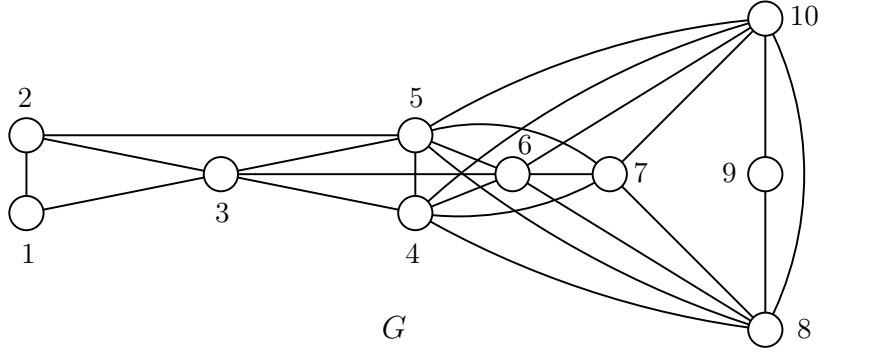


Figure 1: Example of a quasi-line graph G together with a reconstruction (H, \mathbf{H}, ϕ) that uses four subtrees. The fuzzy pairs are $\{1, 4\}$, $\{1, 5\}$, and $\{2, 4\}$. A second example of a reconstruction of G is (H', \mathbf{H}', ϕ') . It uses five subtrees, has no fuzzy pairs and is pleasant (as defined in Section 2).

Related work. Several partial results, proving Opsut’s conjecture for certain graph classes, have previously been obtained. In particular it has been proven for quasi-line graphs for which there are coverings of the neighborhood of all adjacent vertices that are compatible in a certain sense [11]. This result was extended to all non-critical quasi-line graphs [17]. In this context, critical graphs are all quasi-line graphs, in which every clique C has a vertex whose neighbors outside C cannot be covered by a single clique. Subsequently the conjecture has also been proven for bubble-free series-parallel graphs [18]. Opsut’s original paper [15], which contains Conjecture 1, shows an upper and a lower bound in terms of the clique edge cover number. The

clique edge cover number of a graph is the least number of cliques that together cover every edge. Assuming his conjecture, one can use his bounds to deduce that every quasi-line graph on n vertices has a clique edge cover of size at most n . Chen, Jacobson, Kézdy, Lehel, Scheinerman and Wang [1] proved that indeed every quasi-line graph has such an clique edge covering. Thus, by proving Conjecture 1, we obtain an alternative proof for their result.

Concerning Conjecture 2, there are numerous papers that contain partial results. For graphs without holes the conjecture was already proven in the original paper that introduced competition numbers [16]. This result was extended to graphs with at most one hole [2] and then to graphs with at most two holes [12]. The conjecture was also proven for graphs in which the intersection of holes is restricted in a certain way. More specifically the conjecture was proven for graphs in which holes either share at most one vertex, or share an edge and contain at least 5 vertices [13]. It was also proven for graphs with mutually edge disjoint holes [10] and for graphs in which each hole has an edge that is not contained in any other hole [6]. Theorem 24, the stronger form of Conjecture 2, was recently shown to hold for most graph classes for which Conjecture 2 was known to hold [9].

Notation. Throughout the paper we use finite simple graphs that may either have directed edges, which we call arcs, or undirected edges. By $V(G)$ and $E(G)$ we denote vertices and edges of a graph G , respectively. For a subset of vertices $V' \subseteq V(G)$ we let $G[V']$ be the subgraph of G induced by the vertices in V' . For a vertex v in an undirected graph, we let $N_G(v) := \{u \mid \{u, v\} \in E(G)\}$ be the *open neighborhood* of v in G . We omit the index G if the graph is apparent from the context. A *clique* is a set of vertices that induces a complete graph. Recall that a *simplicial vertex* is a vertex whose neighborhood forms a clique. A set of cliques C_1, \dots, C_t covers a set of edges E if for every edge $e \in E$ there is an $i \in \{1, \dots, t\}$ such that e is an edge in the graph induced by C_i .

For $S \subseteq V(G)$, we define $G - S := G[V(G) \setminus S]$ to be the graph obtained by removing vertices in S and all edges incident with a vertex in S . When removing a single vertex v we write $G - v := G - \{v\}$. A *vertex separator* in a graph G is a (possibly empty) subset of the vertices $S \subseteq V(G)$ such that $G - S$ is a disconnected graph.

2 A characterization of quasi-line graphs

In this section we prove Theorem 1 showing a graph has a fuzzy reconstruction if and only if it is a quasi-line graph. When referring to a fuzzy reconstruction, in the rest of the paper, we frequently omit the word fuzzy. When dealing with a reconstruction (H, ϕ, \mathbf{H}) , to improve readability, we use the letters u , v and w to denote vertices of the graph G and we use the letters h and g or expressions of the form $\phi(v)$ to denote vertices of the graph H . Given a reconstruction (H, ϕ, \mathbf{H}) of a quasi-line graph G , we call a pair of distinct vertices $\{v, v'\} \subseteq V(G)$ *fuzzy* if the pair satisfies the assumption of Property 2, i.e, if the two vertices are non-adjacent and there is a tree $T \in \mathbf{H}$ with $\phi(v), \phi(v') \in V(T)$. Note that in this case T is a path with endpoints $\phi(v)$ and $\phi(v')$. With this terminology, if Property 1, 2 and 4 hold, then Property 3 is equivalent to the following statement:

3'. If $\{v_1, v'_1\} \subseteq V(G)$ is a fuzzy vertex pair, then there is exactly one tree $T \in \mathbf{H}$ that contains $\phi(v_1)$ and $\phi(v'_1)$. Furthermore, if $\{v_2, v'_2\} \subseteq V(G)$ is also a fuzzy vertex pair, then the sets $\{\phi(v_1), \phi(v'_1)\}$ and $\{\phi(v_2), \phi(v'_2)\}$ are either equal or disjoint.

We recall several definitions from [3] (fixing a minor typo).

Definition 2 (Chudnovsky, Seymour [3]). A graph G is a *fuzzy circular interval graph* if

- there is a map ϕ from $V(G)$ to a circle C , and
- there is a set of non-trivial closed intervals from C , none including another, and such that no point of C is the end of more than one of the intervals, so that
- for u, v in G , if u, v are adjacent, then $\{\phi(u), \phi(v)\}$ is a subset of one of the intervals, and if u, v are non-adjacent, then $\phi(u), \phi(v)$ are both ends of any interval that includes both of them (and in particular, if $\phi(u) = \phi(v)$, then u, v are adjacent).

Lemma 3. *Every fuzzy circular interval graph has a reconstruction.*

Proof. Let ϕ, C and a set of intervals in C be given that satisfy the properties in the definition of a fuzzy circular interval graph G . Define H to be the graph on the vertex set $\phi(V(G))$ in which two vertices are adjacent, if they are consecutive on the circle C . Let \mathbf{H} be the set of paths of H that are induced by the intersection of $V(H)$ with one of the given intervals.

We claim that the triple (H, \mathbf{H}, ϕ) is a reconstruction of G . Property 1, 2 and 4 follow from the definition of fuzzy circular interval graphs. Property 3 follows from the fact that all intervals have distinct endpoints. \square

Definition 4 (Chudnovsky, Seymour [3]). A graph G with two distinguished vertices a and b in $V(G)$ is a *fuzzy linear interval strip* if

- a and b are simplicial,
- there is a map ϕ from $V(G)$ to a line L ,
- there is a set of non-trivial closed intervals in L , none including another, and such that no point of L is the end of more than one of the intervals, so that
- for u, v in G , if u, v are adjacent, then $\{\phi(u), \phi(v)\}$ is a subset of one of the intervals, and if u, v are non-adjacent, then $\phi(u), \phi(v)$ are both ends of any interval including both of them (and in particular, if $\phi(u) = \phi(v)$, then u, v are adjacent) and
- $\phi(a)$ and $\phi(b)$ are different from $\phi(v)$ for all other vertices v of G .

We refer to the vertices a and b in the definition as ends of the strip, and denote by (G, a, b) the strip with these ends.

Lemma 5. *If G is a fuzzy linear interval strip, then G has a reconstruction (H, \mathbf{H}, ϕ) for which H is a path and the ends do not have the same image under ϕ as any other vertex of G and no end is part of a fuzzy pair.*

Proof. Let ϕ and a set of subintervals of a line L be given that satisfy the properties in the definition of a fuzzy linear interval strip.

Define H to be the graph on the vertex set $\phi(V(G))$ in which two vertices are adjacent, if they are consecutive on the line L . Note that H is a path.

Let \mathbf{H} be the set of subtrees of H that are induced by the intersection of $V(H)$ with one of the given intervals. We claim that the triple (H, \mathbf{H}, ϕ) is a reconstruction of G . Properties 1–4 follow as in the proof of Lemma 3. The fact that the ends do not have the same image under ϕ as any other vertex of G follows from the corresponding additional requirement in the definition of fuzzy linear interval strip.

It remains to fulfill the requirement that ends are not part of fuzzy pairs. Let $a \in V(G)$ be an end of the strip. We alter the reconstruction so that end a is not part of a fuzzy pair.

Repeating the alteration with the other end then yields a reconstruction with the required property. If a is not part of a fuzzy pair, no alteration is required. Thus, suppose $\{a, v\}$ is a fuzzy pair and $T \in \mathbf{H}$ is a tree such that $\phi(a), \phi(v) \in V(T)$. Note that T is a path with distinct endpoints $\phi(a)$ and $\phi(v)$. Since in the fuzzy linear interval representation there is only one interval that has $\phi(a)$ as endpoint, there is only one such tree T . Let $h \in V(T)$ be the neighbor of $\phi(v)$ in T . We subdivide the edge $\{\phi(v), h\}$, adding a new vertex h' , in H as well as in every tree $T' \in \mathbf{H}$ that contains the edge $\{\phi(v), h\}$. We then alter ϕ by setting $\phi(w) = h'$ for all vertices in $w \in \phi^{-1}(\phi(v)) \cap N(a)$. We replace T by two new trees $T_1 := T - \phi(v)$ and $T_2 := T - \phi(a)$. If a tree $T' \in \mathbf{H} \setminus \{T\}$ originally contained $\phi(v)$ but not h we add h' and the edge $\{\phi(v), h'\}$ to T' . This gives a new reconstruction of G in which a is not part of a fuzzy pair. Note that the property that ends do not have the same image under ϕ as any other vertex of G is maintained. \square

Definition 6 (Chudnovsky, Seymour [3]). The *composition of two strips* is defined as follows: Suppose that (G, a, b) and (G', a', b') are two strips. We compose them as follows. Let A, B be the set of vertices of $G - \{a, b\}$ adjacent in G to a, b respectively, and define A', B' similarly. Take the disjoint union of $G - \{a, b\}$ and $G' - \{a', b'\}$ and let H be the graph obtained from this by adding all possible edges between A and A' and between B and B' . Then H is the composition of the two strips.

In general the *composition of strips* is defined as follows: Start with a graph G_0 with an even number of vertices and which is the disjoint union of complete graphs, and pair the vertices of G_0 . Let the pairs be $(a_1, b_1), \dots, (a_n, b_n)$, say. For $i = 1, \dots, n$, let (G'_i, a'_i, b'_i) be a strip. For $i = 1, \dots, n$, let G_i be the graph obtained by composing (G_{i-1}, a_i, b_i) and (G'_i, a'_i, b'_i) ; then the resulting graph G_n is called a composition of the strips (G'_i, a'_i, b'_i) ($1 \leq i \leq n$).

The structure theorem of Chudnovsky and Seymour says that every connected quasi-line graph is a fuzzy circular interval graph, or a composition of fuzzy linear interval strips.

Lemma 7. *A graph that is the composition of fuzzy linear interval strips has a reconstruction.*

Proof. Let G be the graph obtained by composing the fuzzy linear interval strips (G'_i, a'_i, b'_i) with $i \in \{1, \dots, n\}$. By Lemma 5 each strip (G'_i, a'_i, b'_i) has a reconstruction $(H_i, \mathbf{H}_i, \phi_i)$ for which H_i is a path and for which the ends do not have the same image under ϕ_i as any other vertex of H_i . Additionally we can require that the ends are not part of fuzzy vertex pairs. W.l.o.g. all graphs H_i are disjoint. Define a map ψ from the set of ends $\{a'_1, \dots, a'_n, b'_1, \dots, b'_n\}$ to $V(G_0)$ by mapping a'_i to a_i and b'_i to b_i . Let H be the graph obtained by forming the disjoint union of all graphs H_i and then adding all edges $\{u, v\}$, where u and v are distinct elements of $\{a'_1, \dots, a'_n, b'_1, \dots, b'_n\}$ with $\{\psi(u), \psi(v)\} \in E(G_0)$. We let

$$\mathbf{H} := \bigcup_{i=1}^n \{T \in \mathbf{H}_i \mid V(T) \text{ contains no end of } H_i\} \cup \{T_C \mid C \text{ a connected component of } G_0\}$$

where for a component C the graph T_C is defined as some subtree of H spanning all vertices for which there exists a tree T in some \mathbf{H}_i whose vertex set $V(T)$ contains a vertex that maps to C via ψ .

Let $\phi: V(G) \rightarrow V(H)$ be the map given by $\phi(v) = \phi_i(v)$ if $v \in V(G_i)$.

We claim that (H, \mathbf{H}, ϕ) is a reconstruction of G . We first argue that Property 1 holds. For two adjacent vertices v, v' for which $\phi(v)$ and $\phi(v')$ lie in a common graph H_i , there is a tree $T \in \mathbf{H}_i$ that contains $\phi(v)$ and $\phi(v')$. This tree is also contained in \mathbf{H} . Suppose now vertices v, v' are adjacent but $\phi(v)$ and $\phi(v')$ lie in different graphs H_i and H_j . This implies that there are ends w and w' mapping to the same component C of G_0 such that w is adjacent

to v and w' is adjacent to v' . Thus T_C contains $\phi(v)$ and $\phi(v')$. We argue that the remaining properties hold: Property 2 and 3 follow from the fact that for any component C of G_0 all preimages of vertices in T_C are adjacent and from the fact that the properties hold for the reconstructions of the strips. Property 4 follows since the graphs H_i in reconstruction of the strips are paths. \square

Lemma 8. *Every quasi-line graph has a reconstruction.*

Proof. Since we can obtain a reconstruction for a disconnected graph from reconstructions of its components, it suffices to show the lemma for connected graphs. Thus, suppose G is a connected quasi-line graph. By the structure theorem [3] the graph G is a fuzzy circular interval graph, in which case G has a reconstruction by Lemma 3, or a composition of fuzzy linear interval strips, in which case G has reconstruction by Lemma 7. \square

To prove the converse of Theorem 1 we first show that every graph that has a reconstruction also has a reconstruction that satisfies additional properties. We say a reconstruction is *pleasant* if the following five additional properties hold:

5. If $\{v, v'\} \subseteq V(G)$ is a fuzzy vertex pair, then there is a vertex $v'' \in V(G)$ adjacent to v such that $\phi(v'') = \phi(v')$.
6. For every vertex v of G the vertex $\phi(v)$ has degree at most 2 in H .
7. Each tree $T \in \mathbf{H}$ has at least two vertices and for every leaf h of T there is a vertex $v \in V(G)$ with $\phi(v) = h$.
8. $E(H) = \bigcup_{T \in \mathbf{H}} E(T)$.
9. There are no two distinct trees $T_1, T_2 \in \mathbf{H}$ such that $V(T_1) \subseteq V(T_2)$.

Figure 1 shows a reconstruction (H, ϕ, \mathbf{H}) which, due to violations of Properties 5 and 6, is not pleasant. It also shows a pleasant reconstruction (H', ϕ', \mathbf{H}') derived from this. In fact, every reconstruction of a graph G can be altered into a pleasant reconstruction of G .

Lemma 9. *A graph that has a reconstruction also has a pleasant reconstruction.*

Proof. (Property 5.) Suppose $\{v, v'\} \subseteq V(G)$ is a fuzzy vertex pair but there is no vertex $v'' \in V(G)$ adjacent to v for which $\phi(v'') = \phi(v')$. We now construct a different reconstruction of G that has at least one fuzzy vertex pair less than the original reconstruction. For this let $T \in \mathbf{H}$ be such that $\phi(v), \phi(v') \in V(T)$. The properties of a reconstruction imply that T is a path with endpoints $\phi(v)$ and $\phi(v')$. By Property 3', T is the only tree in \mathbf{H} that contains both $\phi(v)$ and $\phi(v')$. Let $h \in V(T)$ be the neighbor of $\phi(v)$ in T . We subdivide the edge $\{\phi(v), h\}$, adding a new vertex h' , in H as well as in every tree $T' \in \mathbf{H}$ that contains the edge $\{\phi(v), h\}$. We then alter ϕ by setting $\phi(w) = h'$ for all vertices in $w \in \phi^{-1}(\phi(v)) \setminus \{v\}$. We replace T by two new trees $T_1 := T - \phi(v)$ and $T_2 := T - \phi(v')$. If a tree $T' \in \mathbf{H} \setminus \{T\}$ originally contained $\phi(v)$ but not h we add h' and the edge $\{\phi(v), h'\}$ to T' . This gives a new reconstruction of G that has one fuzzy pair less. By repeating the modification we eventually obtain a reconstruction that fulfills Property 5.

(Property 6.) We now argue that we can avoid that ϕ maps vertices of G to vertices of H of degree larger than 2. Suppose v is mapped to a vertex of degree larger than 2 in H . By definition, there is at most one tree $T \in \mathbf{H}$ that contains $\phi(v)$. We add a new auxiliary vertex h and the edge $\{h, \phi(v)\}$ to H and to T . For every vertex $w \in \phi^{-1}(\phi(v))$, we then redefine ϕ by setting $\phi(w) := h$. By repeating this operation we obtain a reconstruction that does not map

any vertices of G to vertices of H of degree larger than 3. Property 5 is maintained by the modification.

(Property 7). We first argue that we can find a reconstruction in which for every $T \in \mathbf{H}$ there are vertices v and v' such that $\phi(v), \phi(v') \in V(T)$ and $\phi(v) \neq \phi(v')$. Let $T \in \mathbf{H}$ be a tree that does not have this property. If there is at most one vertex v with $\phi(v) \in V(T)$, then we simply delete T from \mathbf{H} . Suppose otherwise. There is a vertex h of T and two vertices v, v' of G , such that $\phi(v) = \phi(v') = h$. If there is another tree $T' \in \mathbf{H}$ with $h \in V(T')$ we can delete T and still have a valid reconstruction. Otherwise we add a new auxiliary vertex h' to H and add the edge $\{h', h\}$. We then redefine T to be the tree induced by h' and h and redefine $\phi(v)$ to be h' . This gives us a valid reconstruction and the new T has two vertices which are images of vertices of G under ϕ .

To achieve that all leaves of the trees are images of vertices of G , from every tree $T \in \mathbf{H}$ we repeatedly delete leaf vertices $h \in V(T)$ for which no vertex $v \in V(G)$ exists with $\phi(v) = h$. This yields a reconstruction that fulfills Property 7. Note that these modifications of the reconstruction maintain Properties 5 and 6.

(Properties 8 and 9.) By deleting all the edges of H that are not in $\bigcup_{T \in \mathbf{H}} E(T)$ and repeatedly removing trees $T \in \mathbf{H}$ whose vertex set is contained in some other tree $T' \in \mathbf{H}$ we obtain a reconstruction that satisfies Properties 8 and 9 while maintaining Properties 5–7. \square

Let (H, ϕ, \mathbf{H}) be a reconstruction. For every vertex $h \in V(H)$ and every edge $e \in E(H)$ incident with h , let $\mathcal{P}_{h,e}$ be the set of paths P in H that start at h , continue with e as first edge, and for which there exists a tree $T \in \mathbf{H}$ that contains P as a subgraph.

Lemma 10. *Let (H, ϕ, \mathbf{H}) be a pleasant reconstruction. Let $h \in V(H)$ be a vertex and $e \in E(H)$ an edge incident with h .*

1. *There is a unique tree $T_{h,e} \in \mathbf{H}$ that contains all paths in $\mathcal{P}_{h,e}$ as subgraphs.*
2. *If h' is a leaf of $T_{h,e}$ and e' is the edge of $T_{h,e}$ incident with h' , then $T_{h,e} = T_{h',e'}$.*

Proof. By Property 8 of pleasant reconstructions, at least one tree in \mathbf{H} must contain e . It suffices to show the existence of a unique tree that contains all maximal paths in $\mathcal{P}_{h,e}$. We first assume that there are at least two distinct maximal paths in $\mathcal{P}_{h,e}$. Note that any two distinct maximal paths in $\mathcal{P}_{h,e}$ share a vertex that has degree at least 3 in H . Thus, by Property 4, each path in $\mathcal{P}_{h,e}$ is contained in exactly one tree in \mathbf{H} and this tree is the same tree for all paths in $\mathcal{P}_{h,e}$.

Now suppose there is only one maximal path in $\mathcal{P}_{h,e}$. Then, by definition of $\mathcal{P}_{h,e}$, there is a tree that contains this maximal path and therefore contains all paths in $\mathcal{P}_{h,e}$. If $T, T' \in \mathbf{H}$ were two distinct trees with this property, then due to Property 9 the trees T and T' would share a vertex of degree 3. However, this is not the case by Property 4. This shows uniqueness and thus the first part of the lemma.

To show the second part of the lemma, it suffices now to show that all paths in $\mathcal{P}_{h,e}$ are contained in $T_{h',e'}$. Suppose $P \in \mathcal{P}_{h,e}$. Since $T_{h,e}$ is a tree, there are two (possibly equal) paths P_1 and P_2 in $T_{h,e}$ starting at vertex h' with first edge e' that together cover P . By definition both paths are contained in $T_{h',e'}$ and thus P is a subgraph of $T_{h',e'}$. \square

Proof of Theorem 1. One direction of the theorem has already been established by Lemma 8.

For the converse let G be a graph that has a reconstruction. By Lemma 9 the graph G has a pleasant reconstruction (H, \mathbf{H}, ϕ) . Let v be a vertex of G . We show that all edges incident with v can be covered by two cliques.

Since the reconstruction is pleasant, $\phi(v)$ has degree at most 2. If $\phi(v)$ has degree 0, then v is an isolated vertex or the set $\phi^{-1}(\phi(v))$ is a clique that contains all neighbors of v . Suppose now $\phi(v)$ has degree at least 1. If v is contained in a fuzzy pair, let $T \in \mathbf{H}$ be a path such that there is a vertex u non-adjacent to v for which $\phi(v)$ and $\phi(u)$ form the endpoints of T . Otherwise let e be an edge incident with v and let T be the union of all paths in $\mathcal{P}_{h,e}$. We define for v two cliques S_v^1 and S_v^2 in the following way:

We define $S_v^2 := \{v\} \cup (\phi^{-1}(V(T) \setminus \{\phi(v)\}) \cap N(v))$, and we define $S_v^1 := \{v\} \cup (N(v) \setminus S_v^2)$. By definition, the sets S_v^1 and S_v^2 together cover all edges incident with v and it thus suffices to show that they are cliques.

The set S_w^2 forms a clique by Property 2 of reconstructions. To see that the set S_w^1 is also a clique, it suffices to show that every two distinct vertices $w_1, w_2 \in S_w^1 \setminus \{v\}$ are adjacent. If $\phi(w_1) = \phi(v)$ and w_2 were not adjacent to w_1 , then $\{w_1, w_2\}$ would be a fuzzy vertex pair. Then Property 3' would imply $\phi(w_2) = \phi(u) \subseteq V(T) \setminus \{\phi(w)\}$, which implies $w_2 \notin S_v^1$, giving a contradiction. Symmetrically, it follows from $\phi(w_2) = \phi(v)$ that w_1 and w_2 are adjacent. Finally suppose $\phi(w_1)$ and $\phi(w_2)$ are different from $\phi(w)$. Then there are trees $T_1, T_2 \in \mathbf{H} \setminus \{T\}$ such that for $i \in \{1, 2\}$, we have $\{\phi(v), \phi(w_i)\} \subseteq V(T_i)$. Since $\{u, v\}$ is a fuzzy pair, for $i \in \{1, 2\}$ we also have $V(T) \not\subseteq V(T_i)$. Since $V(T_1)$ and $V(T_2)$ cannot share a vertex of degree at least 3 in H (Property 4), either $\phi(w_1), \phi(w_2) \in V(T_1)$ or $\phi(w_2), \phi(w_2) \in V(T_2)$. W.l.o.g. say $\phi(w_2), \phi(w_2) \in V(T_1)$. The pair $\{w_1, w_2\}$ cannot be fuzzy, since the graph T_1 is either not a path or has an endpoint in $V(T)$. Therefore, w_1 and w_2 are adjacent. This shows that S_w^1 is a clique. \square

Corollary 11. *A graph has a pleasant fuzzy reconstruction if and only if it is a quasi-line graph.*

Concerning reconstructions, we remark that as yet another version of Property 3 we could require that no two distinct paths in \mathbf{H} share an endpoint. This alternative is analogous to the fact that in fuzzy circular interval graphs and fuzzy linear interval strips the intervals may not share endpoints. However, the alternative is not compatible with the properties of pleasant reconstructions. Along these lines, there are various properties one might want to add to a pleasant reconstruction. For example one can require that the trees in \mathbf{H} contain at most one vertex of degree 3. For the definition of pleasant reconstruction we have chosen properties that streamline and simplify our proof of Conjecture 1, which follows next.

3 Competition numbers and quasi-line graphs

In this section we show that quasi-line graphs have competition number at most 2.

We use the following characterization of graphs of competition number at most 2. The original general characterization for arbitrary competition numbers, given by Lundgren and Maybee, contained a slight error which was fixed by Kim (see [7]). However, for graphs of competition number at most 2 the original characterization is correct.

Theorem 12 (Lundgren and Maybee [14]). *A graph on n vertices has competition number at most 2 if and only if there exists a clique edge covering C_1, \dots, C_n and an ordering of the vertices v_1, \dots, v_n such that $v_i \notin C_j$ for $i \geq j + 2$.*

We call the set of edges incident with a vertex v the *vertex star* of v . For a graph G , we say a sequence of cliques C_1, \dots, C_t *incrementally covers* a subset of vertices $V' \subseteq V(G)$ if $|V'| \geq t$ and the following holds: The cliques cover all edges incident with V' and for all $i \in \{1, \dots, t-1\}$ the cliques C_1, \dots, C_{i+1} cover vertex stars of at least i distinct vertices in V' .

Theorem 13. *A graph G has competition number at most 2 if and only if there exists a sequence of cliques that incrementally covers $V(G)$.*

Proof. Suppose G is a graph on n vertices. We show that the theorem is equivalent to Theorem 12. That is, we show that there exists a clique edge covering C_1, \dots, C_n and an ordering of the vertices v_1, \dots, v_n such that $v_i \notin C_j$ for $i \geq j + 2$ if and only if there exists a sequence of cliques that incrementally covers $V(G)$.

For the one direction suppose C_1, \dots, C_n is a clique edge covering and v_1, \dots, v_n an ordering of the vertices such that $v_i \notin C_j$ for $i \geq j + 2$. Then C_n, \dots, C_1 incrementally cover $V(G)$.

For the converse, suppose C_1, \dots, C_n is a sequence of cliques that incrementally covers $V(G)$ and v_1, \dots, v_n is an ordering of the vertices of G such that for all $i \in \{1, \dots, n-1\}$ the vertex star of v_i is covered by C_1, \dots, C_{i+1} . For $i \in \{1, \dots, n-2\}$ let $C'_i := C_{n+1-i} \setminus \{v_1, \dots, v_{n-i-1}\}$ and let $C'_{n-1} := C_2$ and $C'_n := C_1$. Further, for $i \in \{1, \dots, n\}$ let $v'_i := v_{n-i+1}$. The sequence of cliques C'_1, \dots, C'_n is a clique edge covering, since for each C'_i we only remove vertices whose vertex stars are already covered by the cliques C_1, \dots, C_{i-1} . Moreover the sequence of cliques C'_1, \dots, C'_n has the property that $v'_i \notin C'_j$ for $i \geq j + 2$. \square

Lemma 14. *If G is a vertex minimal quasi-line graph with $\text{cn}(G) > 2$, then there is no sequence of cliques that incrementally covers a subset of the vertices of G .*

Proof. Suppose G is a quasi-line graph with competition number larger than 2 and there is a sequence of cliques C_1, \dots, C_t that incrementally covers a subset $V' \subseteq V(G)$. If $|V'| > t$, then there is a strict subset $V'' \subsetneq V'$ which is also incrementally covered by C_1, \dots, C_t . Thus, w.l.o.g., we may assume $|V'| = t$. This implies that there is an ordering v_1, \dots, v_t of the set V' such that for $i \in \{1, \dots, t-1\}$ the vertex star of v_i is covered by C_1, \dots, C_{i+1} and such that the star of v_t is covered by C_1, \dots, C_t . Consider the graph $G' := G - \{v_1, \dots, v_t\}$. Since G has minimum size among the quasi-line graphs that have competition number larger than 2 and since induced subgraphs of quasi-line graphs are quasi-line graphs, the graph G' has competition number at most 2. By Theorem 13 there is an incremental clique covering C'_1, \dots, C'_{n-t} of $V(G')$. The sequence of cliques $C_1, \dots, C_t, C'_1, \dots, C'_{n-t}$ forms an incremental clique covering of $V(G)$, and therefore the competition number of G is at most 2, which gives a contradiction. \square

In the rest of this section we show that every quasi-line graph has an incremental clique covering of a subset of the vertices. By the previous lemma this shows that there are no minimal counterexamples to Conjecture 1 and thus no counterexamples at all.

Lemma 15. *If a graph has a pleasant reconstruction that has a fuzzy vertex pair, then there exists an incremental clique covering of a subset of the vertices.*

Proof. Suppose (H, ϕ, \mathbf{H}) is a pleasant reconstruction of a graph G . Further suppose $\{v_1, u_1\}$ is a fuzzy vertex pair and $T \in \mathbf{H}$ contains both v_1 and u_1 . We define for every vertex w with $\phi(w) \in \{\phi(v_1), \phi(u_1)\}$ two cliques S_w^1 and S_w^2 in the following way: We define $S_w^2 := \{w\} \cup (\phi^{-1}(V(T) \setminus \{\phi(w)\}) \cap N(w))$, and we define $S_w^1 := \{w\} \cup (N(w) \setminus S_w^2)$. The set S_w^2 forms a clique by Property 2 of reconstructions. To see that the set S_w^1 is also a clique, it suffices to show that every two distinct vertices $w_1, w_2 \in S_w^1 \setminus \{w\}$ are adjacent. If $\phi(w_1) = \phi(w)$ and w_2 were not adjacent to w_1 , then $\{w_1, w_2\}$ would be a fuzzy vertex pair. Then Property 3' would imply $\phi(w_2) = \phi(u_1) \subseteq V(T) \setminus \{\phi(w)\}$, which implies $w_2 \notin S_w^1$, giving a contradiction. Symmetrically, it follows from $\phi(w_2) = \phi(w)$ that w_1 and w_2 are adjacent. Finally suppose $\phi(w_1)$ and $\phi(w_2)$ are different from $\phi(w)$. Then there are trees $T_1, T_2 \in \mathbf{H} \setminus \{T\}$ such that for $i \in \{1, 2\}$, we have $\{\phi(w), \phi(w_i)\} \subseteq V(T_i)$. Since $\{u_1, v_1\}$ is a fuzzy pair, for $i \in \{1, 2\}$ we also have $V(T) \not\subseteq V(T_i)$. Since $V(T_1)$ and $V(T_2)$ cannot share a vertex of degree at least 3 in H (Property 4), either $\phi(w_1), \phi(w_2) \in V(T_1)$ or $\phi(w_2), \phi(w_2) \in V(T_2)$. W.l.o.g. say $\phi(w_2), \phi(w_2) \in V(T_1)$. The pair $\{w_1, w_2\}$ cannot be fuzzy, since the graph T_1 is either not a path or has an endpoint in $V(T)$. Therefore, w_1 and w_2 are adjacent. This shows that S_w^1 is a clique.

We claim that $S_w^1 = S_{w'}^1$ if $\phi(w) = \phi(w')$: Indeed, for every vertex in $v \in S_w^1$ either $\phi(v) = \phi(w)$, in which case $v \in S_{w'}^1$, or there is a tree T' different from T with $\{\phi(w), \phi(v)\} \subseteq V(T')$. Since $\{u_1, v_2\}$ is a fuzzy pair, by Property 3' the pair $\{v, w'\}$ is not fuzzy. Therefore v and w' are adjacent, and thus $v \in S_{w'}^1$.

We also claim that for every $u \in \phi^{-1}(\phi(u_1))$ and $x \in N(u)$, either $\{u, x\} \subseteq S_u^1 = S_{u_1}^1$, or there is a $v \in \phi^{-1}(\phi(v_1))$ such that $\{u, x\} \subseteq S_v^2$: Indeed, if $x \notin S_u^1$, then $\phi(x) \in V(T) \setminus \{\phi(u)\}$. If $\phi(x) = \phi(v_1)$ then, we can choose v as x . Otherwise let v be a vertex adjacent to u with $\phi(v) = \phi(v_1)$. This vertex exists by Property 5 of a pleasant reconstruction and $\{u, x\} \subseteq S_v^2$ holds by definition of S_v^2 .

Since the reconstruction is pleasant, by Property 5, the preimages $\phi^{-1}(\phi(v_1))$ and $\phi^{-1}(\phi(u_1))$ have size at least 2. Suppose $\phi^{-1}(\phi(v_1)) = \{v_1, \dots, v_t\}$. By our previous two claims, the clique sequence $S_{v_1}^1, S_{v_1}^2, S_{v_2}^2, \dots, S_{v_t}^2, S_{u_1}^1$ incrementally covers the set $\phi^{-1}(\phi(v_1)) \cup \phi^{-1}(\phi(u_1))$. Thus, by Lemma 14, the graph G is not vertex minimal with $\text{cn}(G) > 2$. \square

Lemma 16. *Let G be a quasi-line graph without simplicial vertices and (H, ϕ, \mathbf{H}) a pleasant reconstruction of G without fuzzy pairs. It is possible to associate with every vertex $v \in V(G)$ a pair of cliques C_v^1, C_v^2 , such that the following holds:*

- For every vertex $v \in V(G)$ the cliques C_v^1 and C_v^2 cover all edges incident with v .
- For every $v \in V(G)$ and every $j \in \{1, 2\}$ there is $v' \neq v$ and $j' \in \{1, 2\}$ such that $C_{v'}^{j'} = C_v^j$.

Proof. Note that, since the reconstruction does not have fuzzy vertex pairs, any set that is of the form $\phi^{-1}(V(T))$, with $T \in \mathbf{H}$, forms a clique in G .

We now argue that for every $v \in V(G)$ the degree of $\phi(v)$ in H is exactly 2. If this is not the case, then, by Property 6 of a pleasant reconstruction, the degree of $\phi(v)$ is smaller than 2. If $\phi(v)$ has degree 1 and is incident with edge e' , then $\phi^{-1}(V(T_{\phi(v), e'}))$ is a clique that covers the vertex star of v and thus v is simplicial. If $\phi(v)$ has degree 0, then, by Property 7, v is of degree 0 and thus simplicial.

We associate two cliques C_v^1, C_v^2 with every vertex $v \in V(G)$. Since $\phi(v)$ has degree exactly 2, there are two edges e', e'' incident with $\phi(v)$. We define $C_v^1 := \phi^{-1}(V(T_{\phi(v), e'}))$ and $C_v^2 := \phi^{-1}(V(T_{\phi(v), e''}))$.

To show the first claim of the lemma suppose v and v' are adjacent in G . We show that one of the cliques C_v^1 or C_v^2 contains v' . If $\phi(v) = \phi(v')$, then $v' \in C_v^1$. Otherwise, by definition of the reconstruction, there is a tree $T \in \mathbf{H}$ that contains both v and v' . Thus there is a path from $\phi(v')$ to $\phi(v)$ that lies entirely in T . Let e' be the edge of this path incident with $\phi(v)$, then the path is contained in $T_{\phi(v), e'}$. Thus one of the cliques C_v^1 or C_v^2 contains v' .

To show the second claim suppose v is a vertex of G and suppose $C \in \{C_v^1, C_v^2\}$. There is an edge e incident with $\phi(v)$ such that $C = \phi^{-1}(V(T_{\phi(v), e}))$. Let $v' \in V(G)$ be a vertex for which $\phi(v')$ is a leaf of $V(T_{\phi(v), e})$ such that $\phi(v') \neq \phi(v)$. Such a vertex exists by Property 7. Let e' be the edge of $T_{\phi(v'), e}$ incident with $\phi(v')$. By Lemma 10, $T_{\phi(v), e} = T_{\phi(v'), e'}$ and we conclude $C = \phi^{-1}(V(T_{\phi(v'), e'})) \in \{C_{v'}^1, C_{v'}^2\}$. \square

Lemma 17. *For every quasi-line graph G there is an incremental clique covering of a subset of the vertices.*

Proof. Since for any simplicial vertex v the clique $N(v) \cup \{v\}$ incrementally covers $\{v\}$, we can assume that G does not have simplicial vertices. Furthermore, by Lemmas 14 and 15, it suffices to show the statement for quasi-line graphs which have a pleasant reconstruction (H, ϕ, \mathbf{H}) without fuzzy pairs. For this reconstruction we can apply Lemma 16, and associate every vertex v with two cliques C_v^1 and C_v^2 .

We consider the following bipartite graph: One bipartition class consists of all vertices $v \in V(G)$. The other bipartition class consists of all cliques C of G for which there exists a vertex $v \in V(G)$ and a $j \in \{0, 1\}$ such that $C = C_v^j$. The edge set is $\{\{v, C_v^i\} \mid v \in V(G), i \in \{1, 2\}\}$.

This bipartite graph has minimum degree 2, since every vertex is adjacent to two cliques and since by Lemma 16 every clique is adjacent to at least two vertices. Let $C_1, v_1, \dots, C_t, v_t$ be a cycle in the bipartite graph. By construction, for every $k \in \{1, \dots, t-1\}$, the cliques C_k and C_{k+1} cover the vertex star of v_k . Moreover C_1 and C_t cover the vertex star of v_t . Thus the sequence C_1, \dots, C_t incrementally covers the set $\{v_1, \dots, v_t\}$. \square

Theorem 18. *If G is quasi-line graph, then G has competition number at most 2.*

Proof. By Lemma 17 every quasi-line graph G has an incremental clique covering of a subset of the vertices. Lemma 14 then shows that there are no vertex minimal quasi-line graphs with competition number greater than 2, and thus all quasi-line graphs have competition number at most 2. \square

Corollary 19 (Chen, Jacobson, Kézdy, Lehel, Scheinerman, Wang [1]). *Every quasi-line graph on n vertices has a clique edge covering consisting of n cliques.*

4 Competition numbers and holes

To study the relationship between the competition numbers and the number of holes in a graph, we introduce some additional terminology. Recall that a hole in a graph is a subset of at least 4 vertices that induces a simple cycle. We denote by $\text{hole}(G)$ the number of holes in a graph G and by $\text{hole}(v)$ the number of holes containing the vertex v . For a graph G , we let $\omega(G)$ be the size of a largest clique of G . For a vertex v in a graph G we define its *simplicial defect* as $\text{def}(v) := |N(v)| - \omega(G[N(v)])$. The terminology is justified because a vertex is simplicial if and only if its defect is 0.

We say a vertex v is *good* if for every two non-adjacent vertices $u, u' \in N(v)$ there exists a hole that contains v, u and u' . Note that such a hole cannot contain any other vertices from $N(v)$ besides u and u' . This implies that $\text{def}(v) \leq \text{hole}(v)$ for every good vertex v .

Next we show that every graph has a good vertex. The proof is a generalization of a proof of Dirac [5] that shows that every non-complete chordal graph has two non-adjacent simplicial vertices.

Lemma 20. *If G is a non-complete graph, then G contains two non-adjacent good vertices.*

Proof. We show the statement by induction on the number of vertices of G .

Let G be a smallest non-complete graph which is a counterexample to the lemma, and let S be a minimal vertex separator (which is empty in the case that G is disconnected). Let (A, B) be a non-trivial partition of the vertices in $G - S$ such that there is no edge of G with an endpoint in A and an endpoint in B . Consider the graph G_A which is obtained in the following way: Start with the subgraph of G that is induced by the vertex set $A \cup S$, then add all edges between vertices of S , as detailed in Figure 2. (Be aware that G_A is not necessarily an induced subgraph of G .)

We next note that G_A has a good vertex v in A . From the definition, this is equivalent to showing that every path u, v, u' of distinct vertices lies on an induced cycle (perhaps a triangle). Obviously, every vertex of G_A is good if G_A is a complete graph. Otherwise, by the induction hypothesis, G_A has two non-adjacent good vertices. Since S is a clique of G_A , by construction, at least one of the good vertices is in A .

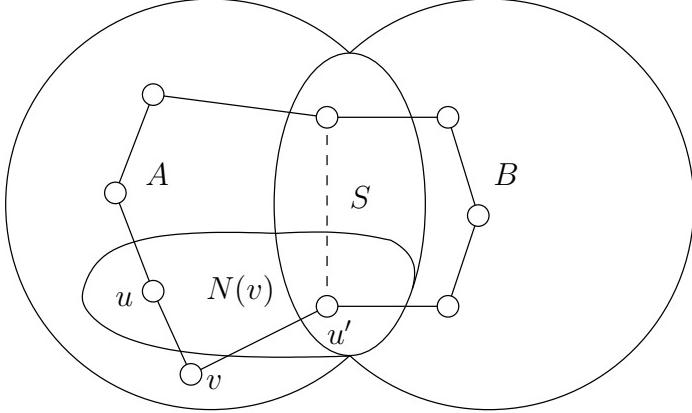


Figure 2: The figure illustrates the construction of G_A used in the proof of Lemma 20. Edges within the vertex separator S are added (dashed line) and G_A is then obtained as the subgraph induced by $A \cup S$. The figure also illustrates how, in the proof, the induced cycle C in G_A is altered to obtain an induced cycle in G which contains v , u and u' .

We now show that v is also good in G . Let u, u' be distinct neighbors of v in G . Since v is good in G_A , there is an induced cycle C in G_A that includes $\{v, u, u'\}$. Since the cycle is induced, C contains at most 2 vertices of S . This means that C is also an induced cycle in G unless $C \cap S$ is an edge $\{x, y\}$ that is not an edge of G . In that case, if x was not adjacent to every component of $G - S$, then $S - x$ would be a smaller separator, and similarly for y , so there must be a path P in G from x to y whose internal vertices all lie in B . Choosing such a path of minimal length, we find that $C \cup V(P)$ is an induced cycle in G .

Thus v is a vertex in A that is good in the graph G . By symmetry there is also a good vertex in B . Since vertices in A are not adjacent to vertices in B , this concludes the proof. \square

Corollary 21. *Every graph has a good vertex.*

Proof. If a graph is complete, then every vertex is good. If a graph is not complete, it contains a good vertex by Lemma 20. \square

Lemma 22. *If v is a vertex of a graph G on at least 2 vertices, then $\text{cn}(G) \leq \max\{1, \text{cn}(G - v)\} + \text{def}(v)$.*

Proof. Let $t := \max\{1, \text{cn}(G - v)\}$. By definition, there exists a directed acyclic graph D_v with competition graph $(G - v) \dot{\cup} t \cdot K_1$, where $t \cdot K_1$ is an empty graph on t vertices. By possibly deleting unnecessary arcs, we can achieve that D_v has a vertex $u \in V(t \cdot K_1)$ that does not have any outgoing arcs. Moreover, we can require that $v \notin V(t \cdot K_1)$, implying that $u \notin V(G)$.

We design a directed acyclic graph D defined on the vertex set $V(D_v) \dot{\cup} \{v\} \dot{\cup} \{v_1, \dots, v_{\text{def}(v)}\}$ such that $G \dot{\cup} t \cdot K_1 \dot{\cup} \text{def}(v) \cdot K_1$ is the competition graph of D . Its existence implies $\text{cn}(G) \leq t + \text{def}(v)$ and thus the theorem.

To design D , we start with D_v . To the vertex set of D_v , we add the set $\{v, v_1, v_2, \dots, v_{\text{def}(v)}\}$. Let C be a largest clique in $G[N(v)]$ and let $\{z_1, \dots, z_{\text{def}(v)}\} := N(v) \setminus C$ be the set of neighbors of v not contained in C .

We first describe the set of arcs of D informally. We redirect all arcs that previously ended in u to now end in v . Next we add all arcs (x, u) with $x \in C$ and the arc (v, u) . This is possible since $u \notin \{v\} \cup C$. For each $i \in \{1, \dots, \text{def}(v)\}$ we add the arcs (v, v_i) and (z_i, v_i) .

More formally, D is the graph on the vertex set $V(D_v) \dot{\cup} \{v\} \dot{\cup} \{v_1, \dots, v_{\text{def}(v)}\}$. Its set of arcs is formed as follows: We set

$$\begin{aligned} E_1 &:= \{(y, u) \mid y \in N_{D_v}(u)\} \\ E_2 &:= \{(y, v) \mid y \in N_{D_v}(u)\} \\ E_3 &:= \{(x, u) \mid x \in C\} \cup \{(v, u)\} \\ E_4 &:= \{(v, v_i), (z_i, v_i) \mid i \in \{1, \dots, \text{def}(v)\}\}, \end{aligned}$$

and we define the arcs of D as $E(D) := (E(D_v) \setminus E_1) \cup E_2 \cup E_3 \cup E_4$.

The graph D is acyclic: Prior to our first modification of the arcs, v is an isolated vertex and u has no outgoing arcs. Thus our first modification, i.e., removing the set E_1 and adding the set E_2 , redirects the arcs ending in u to end in v . This can be interpreted as swapping the names of u and v and thus does not introduce cycles. By adding the arcs sets E_3 and E_4 , we exclusively add arcs that end in a vertex in $\{u, v_1, \dots, v_{\text{def}(v)}\}$. However, no vertex in $\{u, v_1, \dots, v_{\text{def}(v)}\}$ has outgoing arcs, and thus no arcs in E_3 or E_4 can be part of a cycle. This shows that D is acyclic.

By construction $G \dot{\cup} t \cdot K_1 \dot{\cup} \text{def}(v) \cdot K_1$ is the competition graph of D and thus we conclude that G has competition number at most $t + \text{def}(v)$, which concludes the theorem. \square

Lemma 22 and Corollary 21 allow us to prove the conjecture of Kim [8] concerning the relation between the competition number and the number of holes in a graph.

Theorem 23. *If a graph has at most k holes, then it has competition number at most $k + 1$.*

Proof. We show the statement by induction on n , the number of vertices of G . For $n = 1$ the statement is obvious. Now suppose $n \geq 2$. By Corollary 21 the graph G contains a good vertex v . Since $\text{hole}(G) = \text{hole}(G - v) + \text{hole}(v)$, it suffices to show that $\text{cn}(G) \leq \max\{1, \text{cn}(G - v)\} + \text{hole}(v)$. Since v is good and therefore $\text{def}(v) \leq \text{hole}(v)$ the theorem follows from Lemma 22. \square

For a graph G , the *cycle space* is the \mathbb{F}_2 vector space where the vectors are those sets of edges of G which form a subgraph that has only vertices of even degree. Addition in this vector space is defined as the symmetric difference of the edge sets. Every hole is a vector in the cycle space. For a graph G let the hole space $\mathcal{H}(G)$ be the subspace of the cycle space spanned by all the holes. For various graphs for which Theorem 23 was previously known, a stronger upper bound for the competition number in terms of the dimension of the hole space has recently been proven [9]. We extend this stronger upper bound to all graphs.

Theorem 24. *For any graph G , we have $\text{cn}(G) \leq \dim(\mathcal{H}(G)) + 1$.*

Proof. We show the statement by induction on n , the number of vertices of G . For $n = 1$ the statement is obvious. Now suppose $n \geq 2$. By Lemma 22, $\text{cn}(G) \leq \max\{1, \text{cn}(G - v)\} + \text{def}(v)$ for every vertex v . It thus suffices to argue that $\dim(\mathcal{H}(G)) \geq \dim(\mathcal{H}(G - v)) + \text{def}(v)$ for every good vertex v . Let C be a largest clique of $N(v)$. Since C is maximal, for every vertex $z \in N(v) \setminus C$ there is a non-adjacent vertex $u \in C$. Since v is good, it then follows that for every vertex $z \in N(v) \setminus C$ there exists a hole H_z which contains v and z and no other vertices from $N(v) \setminus C$. Let \mathcal{B} be a basis for the hole space $\mathcal{H}(G - v)$, and consider the set $\mathcal{B}' := \mathcal{H}(G - v) \cup \{H_z \mid z \in N(v) \setminus C\}$. Every hole H_z has an edge, namely $\{v, z\}$, that is not contained in any other hole in \mathcal{B}' . Thus \mathcal{B}' is an independent set and $\dim(\mathcal{H}(G)) \geq \dim(\mathcal{H}(G - v)) + |N(v) \setminus C| = \dim(\mathcal{H}(G - v)) + \text{def}(v)$. \square

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